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The Abstract Definitions of Groups of Degree 8.

BY JOSEPHINE E. BURNS.

INTRODUCTION.

Abstract group theory may be said to begin in the work of Cayley. In 1854 he first defined a group by means of relations existing among its operators. In two papers published in the *Philosophical Magazine*[*] he defined a few elementary groups by means of such abstract relations. In a sequel to these papers which appeared in 1859, he defined abstractly the five groups of order 8 and the system of dihedral groups. On the other hand, Cayley did not attempt to formulate a general abstract definition of a group until several years later (1878), not, in fact, until after such a formulation had been made by others. From the number of pioneers in abstract group theory the name of Sir William Hamilton should not be omitted; for to him are due the first definitions of the important category of groups known as the groups of the regular polyhedrons. These were included in an article published in the *Proceedings of the Royal Irish Academy*[†] in 1856, just two years after Cayley's first articles. However, in spite of these beginnings on English soil, the first actual attempts at an abstract definition of a group were found in the work of a German mathematician some fifteen years later. Kronecker,[‡] in 1870, gave a really abstract definition of a group, although he considered only the case in which the operators were commutative. Dyck, in 1882, published quite an extensive article in the *Mathematische Annalen,* in which he made marked advance over anything which had previously appeared. He explicitly defined the simple group of order 168, and a group of order $2mn$ defined by the relations

$$s_1^2 = s_2^2 = s_3^2 = s_4^2 = s_1 s_2 s_3 s_4 = (s_1 s_2)^m = (s_3 s_2)^n = 1.$$

The contributions of Weber (1882) and of Frobenius (1887) should also be noted in this connection.

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[*] "Papers," 2 (1854), p. 123.

[†] *Proceedings of the Royal Irish Academy,* 6 (1853–1857), p. 415.

[‡] Kronecker, "Werke," 1, p. 274.

Netto* was the first to examine the possible orders of groups which may be generated by operations satisfying certain defining relations. He sought the number of possible products of s_1 and s_2 when they are connected by the equation $s_1 s_2 = s_2 s_1^m$. He also enunciated a theorem in regard to the order of the group generated by three operators satisfying quite elementary conditions. In recent years, much has been done by Professor G. A. Miller in regard to the groups generated by a small number of operators satisfying simple equations of relation. In particular, his article entitled "The Abstract Definitions of All the Substitution Groups Whose Degree Does not Exceed Seven"† forms the starting-point for the present paper.

The first part of this paper is devoted to the proof of a few general theorems relative to the groups generated by two operators satisfying certain defining relations. The second part consists of the abstract definitions of the substitution groups of degree 8, including applications of the theorems of the first part.

I. GENERAL THEOREMS.

In the theorems which follow, the equation $s^n = 1$ implies that s is of exactly order n, and when relations are given connecting two operators it is supposed that no other relations exist except those stated and such others as may be derivable from them.

1. *The Group Generated by Two Operators Satisfying the Conditions* $s_1^4 = s_2^4,\ (s_1 s_2)^2 = 1,\ s_1^{-1} s_2 s_1 = s_2^{\beta} s_1^{\alpha}$.

The groups generated by two operators s_1 and s_2 satisfying the relations $s_1^4 = s_2^4$, $(s_1 s_2)^2 = 1$, $s_1^{-1} s_2 s_1 = s_2^{\beta} s_1^{\alpha}$ depend upon the congruences of α and β, mod 4. Four cases may be distinguished according as $\beta \equiv 0, 1, 2$ or 3, mod 4. In each case the values of α reduced by the same modulus will give subcases.

CASE 1. $\beta \equiv 0$, mod 4. The third equation in this case reduces to $s_2 = s_1^{\alpha + 4k}$. The group is then cyclic and the order depends upon α. The largest possible group arises when $\alpha \equiv 3$, mod 4, in which case G is of order 8. All other values of α give subgroups of this cyclic group.

CASE 2. $\beta \equiv 1$, mod 4. The third equation now becomes $s_1^{-1} s_2 s_1 = s_2 s_1^{\alpha} \cdot s_2^{\beta - 1} = s_2 s_1^{\alpha} s_2^{4k'}$, where k' is integral. The order of G again depends upon α. The second equation gives the relations

$$s_1^{-2} s_2^{-1} = s_2^{4k+1}, \quad s_1^{-2} = s_2^{4k+2}, \quad s_2^8 = 1.$$

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* Netto, "Substitutionentheorie" (1882), p. 37.

† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), p. 363.

Hence, the orders of s_1 and s_2 divide 8. If this is exactly the order, two possibilities arise. If k is odd, G is of order 16, and may be generated by the two substitutions $s_1 = adcfehgb$, $s_2 = abcdefgh$. In the table of abstract definitions of groups of degree 8, which appears as Part II of the present paper, this is G_{16}, No. 8. If k is even, G is abelian of order 16 and type $(3, 1)$. It may be generated by the two substitutions $s_1 = abcdefgh \cdot ij$, $s_2 = ahgfedcb$. If $s_1^4 = s_2^4 = 1$, G is abelian of order 8 and type $(2, 1)$. From smaller possible orders of s_1 and s_2 arise the four-group, the octic group and the group of order 2.

CASE 3. $\beta \equiv 2$, mod 4. The third equation now becomes $s_1^{-1} s_2 s_1 = s_2^2 s_1^{\alpha} s_2^{4k'}$. If $\alpha \equiv 0$, mod 4, the order of s_1 reduces to 2, and s_2 must be identity. If $\alpha \equiv 1$, mod 4, the relations follow:

$$s_1^{-1} s_2 s_1 = s_2^2 s_1 \cdot s_1^{4k}, \;\; s_1^{-1} = s_2^{4k+1}, \;\; (s_1 s_2)^2 = (s_2^{-4k})^2 = s_2^{-8k} = 1; \;\; s_1^4 = s_2^{-4}, \;\; s_2^8 = 1, \;\; s_1^8 = 1.$$

The group is then the cyclic group of order 8. If $\alpha \equiv 2$, mod 4, $s_1^4 = s_2^2 = 1$ and $s_1^2 = s_2$; hence, G is the cyclic group of order 4. If s_2 reduces to the identical operation, the conditions are satisfied and G is of order 2. If $\alpha \equiv 3$, mod 4, the conditions are contradictory unless $s_1^2 = s_2^2 = 1$ and $s_1 = s_2$.

CASE 4. $\beta \equiv 3$, mod 4. Equation three now becomes $s_1^{-1} s_2 s_1 = s_2^3 s_1^{\alpha} \cdot s_2^{4k'}$. If $\alpha \equiv 0$, mod 4, the following equations result:

$$s_1^{-1} s_2 s_1 = s_2^{4k'+3}, \;\; s_1^{-2} s_2^{-1} = s_2^{4k'+3}, \;\; s_1^{-2} = s_2^{4k}, \;\; s_1^4 = s_2^{-4k} = s_2^4, \;\; s_2^{4k+4} = 1.$$

Now, $s_1 s_2$ and $s_2 s_1$ have a common square. Hence, the product of one into the inverse of the other is transformed into its inverse by each operation.

$$(s_1 s_2^2 s_1)^2 = s_1 s_2^2 s_1^2 s_2^2 s_1 = s_1 s_2^4 s_1 s_1^2 = s_1^8.$$

Hence, s_1^8 both is invariant and is transformed into its inverse by $s_1 s_2$ and $s_2 s_1$. The order of s_1 and s_2 divides 16. We then have as the only possibilities $s_1^2 = s_2^4 = (s_1 s_2)^2 = 1$, $s_1^{-1} s_2 s_1 = s_2^8$, which define the octic group, or $s_1^2 = s_2^2 = (s_1 s_2)^2 = 1$, which define the four-group.

If $\alpha \equiv 1$, mod 4, the conditions are contradictory unless $s_1 = 1$ and $s_2^2 = 1$. If $\alpha \equiv 2$, mod 4, the following conditions arise:

$$(s_1 s_2^2 s_1)^2 = s_1 s_2^2 s_1^2 s_2^2 s_1 = s_2^2 s_1^3 s_2^2 s_1 \cdot s_1^{4k} = s_2^2 s_1^3 \cdot s_1 s_2^2 = s_2^8, \text{ since } s_1^{-1} s_2 s_1 = s_2^3 s_1^2 \cdot s_1^{4k}.$$

The order of s_1 and s_2 then divides 16. But no groups exist for the maximum values of the orders of s_1 and s_2. If $s_1^8 = s_2^8 = 1$, the conditions result:

$$s_2^{-1} s_1^2 s_2 = s_2^3 s_1^2 s_2 s_1^4 = s_1^{-2} s_2^{-1} s_2 \cdot s_1^{4k-4} = s_1^2 s_1^{-4k}.$$

If k is even, s_1^2 and s_2^2 are invariant in G and generate the central. $[s_2, s_1^2]$ is abelian of order 16, type $(3, 1)$. This subgroup is invariant under s_1. G is of order 32. Its existence is proved, since it may be generated by two operators $s_1 = adc'f'ehg'b' \cdot a'd'cfe'h'gb$ and $s_2 = abcdefgh \cdot a'b'c'd'e'f'g'h'$. If k is odd, s_1^2 and s_2^2 are commutative, each is of order 4 and they generate an invariant subgroup of order 8, which is abelian, of type $(2, 1)$. s_2 transforms this H_8 into itself, giving a subgroup of order 16, H_{16}, which is in turn invariant under s_1. G is then of order 32. Its existence is established by the substitutions $s_1 = abchefgd$, $s_2 = abcdefgh$. In the table of definitions of groups of degree 8, this group is No. 9, of order 32.

If $s_1^4 = s_2^4 = 1$, then s_1^2 and s_2^2 are invariant and generate the four-group. Extending this by s_2 gives an abelian subgroup H_8, type $(2,1)$, which is invariant under s_1. G is therefore of order 16. It may be generated by $s_1 = abef \cdot cdgh$, $s_2 = abcd \cdot efgh$. In the table this group is No. 5 of order 16. Degenerate cases are given if $s_1^4 = s_2^4 = 1$, the octic group, or if $s_1^2 = s_2^2 = 1$, the four-group.

If $\alpha \equiv 3$, mod 4, the conditions are contradictory unless s_2 is of order 2 and s_1 reduces to the identical operation.

The preceding results may be collected in the following theorem:

If two operators s_1 *and* s_2 *satisfy the three conditions* $s_1^4 = s_2^4$, $(s_1 s_2)^2 = 1$, $s_1^{-1} s_2 s_1 = s_2^{\beta} s_1^{\alpha}$, *they generate one of eleven groups, two of which are of order* 32, *one of order* 16 *and the others subgroups of these three. Six are non-abelian; and five are abelian, three cyclic and two non-cyclic abelian.*

2. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^4 = s_2^4, \quad (s_1 s_2)^2 = 1, \quad s_1^{-1} s_2 s_1 = s_1^{\alpha} s_2^{\beta}$.

When $s_1^4 = s_2^4$, $(s_1 s_2)^2 = 1$, $s_1^{-1} s_2 s_1 = s_1^{\alpha} s_2^{\beta}$, four cases are possible, according as $\alpha \equiv 0, 1, 2$ or 3, mod 4. Within each case are considered the groups which arise when $\beta \equiv 0, 1, 2$ or 3, mod 4.

CASE 1. $\alpha \equiv 1$, mod 4. If $\beta \equiv 0$, mod 4, the third equation reduces to equation $s_1^{-1} s_2 s_1 = s_1^{4k+1}$. Hence, G must be cyclic and its order can evidently not exceed 4. If $\beta \equiv 1$, mod 4, the conditions are contradictory except for the trivial case $s_1 = s_2^2 = 1$. If $\beta \equiv 2$, mod 4, equation three becomes $s_1^{-1} s_2 s_1 = s_1 s_2^2 s_2^{4k}$ or $s_2 s_1 = s_1^2 s_2^2 \cdot s_2^{4k}$. Then

$$s_1^{-1} s_2^{-1} = s_1^2 s_2^2 \cdot s_2^{4k}, \quad s_1^{-3} s_2^{-3} = s_1^{4k}, \quad s_1 s_2 = s_1^{4k+3}, \quad s_2 = s_1^{4k+7}.$$

The group is then evidently cyclic and the order of s_1 is limited to 8. G is therefore the cyclic group of order 8 or a subgroup. If $\beta \equiv 3$, mod 4, the conditions can be satisfied if and only if $s_1 = s_2^2 = 1$.

CASE 2. $\alpha \equiv 2$, mod 4. If $\beta \equiv 0$, mod 4, the conditions are satisfied only if $s_1^2 = s_2 = 1$. If $\beta \equiv 1$, mod 4, the conditions readily reduce to $s_1^2 = s_2^2 = (s_1 s_2)^2 = 1$

and the four-group results. If $\beta\equiv 2$, mod 4, the conditions again degenerate to the trivial case $s_1^2=s_2=1$. If $\beta\equiv 3$, mod 4, the three defining relations cease to be independent. The third equation may be deduced from the second. The orders of s_1 and s_2 can not exceed 8. The equations then become

$$s_1^8=s_2^8=1, \quad s_1^4=s_2^4, \quad (s_1s_2)^2=1, \quad \text{and} \quad s_1^4=s_2^4=(s_1s_2)^2=1.$$

It is well-known that these conditions are not in themselves sufficient to define groups of finite order. In the trivial case $s_1^4=s_2^2=1$, the octic group is defined and, if $s_1^2=s_2^2=1$, the four-group.

CASE 3. $\alpha\equiv 3$, mod 4. If $\beta\equiv 0$, mod 4, equation three reduces to $s_1^{-1}s_2s_1=s_1^3s_2^{4k}$ or $s_1=s_2^{4k+3}$. Hence, G is cyclic and its order can evidently not exceed 8. All such groups are then subgroups of the cyclic group of order 8. If $\beta\equiv 1$, mod 4, the group is again cyclic and of order 4 or 2. The cases $\beta\equiv 2$ and $\beta\equiv 3$, mod 4, also give rise to these two cyclic groups and only these.

CASE 4. $\alpha\equiv 0$, mod 4. Condition three now becomes $s_1^{-1}s_2s_1=s_1^{4k'}$, $s_2^{\beta}=s_2^{4k+\beta}$. If $\beta\equiv 0$, mod 4, the only conditions possible are evidently $s_1^2=s_2=1$. If $\beta\equiv 1$, mod 4, the following equations result;

$$s_1^{-1}s_2s_1=s_2^{4k+5}=s_1^{-2}s_2^{-1}, \quad s_1^{-2}=s_2^{4k+6}, \quad \therefore\ s_1^6=s_2^6=1.$$

If k is odd, the group is abelian of order 8 and type (2, 1). If k is even, $s_1^{-2}=s_2^6$, $s_1^2=s_2^3$ and $s_1^{-1}s_2s_1=s_2^5$. The result is the group of order 16 containing a cyclic subgroup of order 8, each operator of which is transformed into its fifth power. In the table this group is No. 9 of order 16. If $\beta\equiv 2$, mod 4, the equations again degenerate so that only a group of order 2 results. If $\beta\equiv 3$, mod 4, s_1 and s_2 generate either the octic group or the four-group.

These results may be collected in the theorem:

If two operators s_1 *and* s_2 *satisfy the three conditions* $s_1^4=s_2^4$, $(s_1s_2)^2=1$, $s_1^{-1}s_2s_1=s_1^{\alpha}s_2^{\beta}$, *they generate one of seven groups, two of which are the octic and* G_{16}, *containing an operator of order* 8 *which is transformed into its fifth power. All others are subgroups of* G_{16} *and are abelian. The case in which* $\alpha\equiv 2$ *and* $\beta\equiv 3$, *mod* 4, *is excepted, for when these conditions are fulfilled,* s_1 *and* s_2 *may generate a group of infinite order.*

3. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^4=s_2^4,\quad (s_1s_2)^2=1,\quad s_2^2s_1^2=s_1^2s_2^2$.

In order to find the largest possible orders for s_1 and s_2 when they satisfy the above conditions, the following theorem is applied:

If two operators have a common square, the product of one and the inverse of the other is transformed into its inverse by each operator.

s_1s_2 and s_2s_1 being of order 2, their product is transformed into its inverse by each operator. $(s_1s_2^2s_1)^2 = s_1s_2^2s_1^2s_2^2s_1 = s_1^8$. Hence, the order of s_1 and s_2 divides 16. The possible cases then are

$$1)\ s_1^{16}=s_2^{16}=1,\quad 2)\ s_1^8=s_2^8=1,\quad 3)\ s_1^4=s_2^4=1,\quad 4)\ s_1^2=s_2^2=1,\quad 5)\ s_1s_2=s_2s_1.$$

In the last case the groups are abelian.

CASE 1. $s_1^{16}=s_2^{16}=1$. s_1^4 is an invariant operator of order 4. s_1^4, $s_1s_2^2s_1$, $s_1^2s_2^2$ generate an abelian group of order 16 and type (2,1,1) which is invariant under s_1 and s_2.

$$(s_1^2s_2^2)^{-1}s_1s_2^2s_1(s_1^2s_2^2)=s_1^3s_2^2s_1^3s_2^2s_1^3s_2^2=s_1^8s_1^2s_2s_1^{-2}s_2s_1^2=s_1^4s_1^2s_2s_1^2s_2s_1^2=s_1^4s_1s_2^{-2}s_1=s_1s_2^2s_1,$$
$$s_1^{-1}\cdot s_1s_2s_1\cdot s_1=s_2^2s_1^2,\quad s_1^{-1}s_1^2s_2^2s_1=s_1s_2^2s_1,\quad s_2^{-1}\cdot s_2^2s_1^2 . s_2=s_2s_1^2s_2=s_1s_2^2s_1s_1^8,$$
$$s_2^{-1}\cdot s_1s_2^2s_1\cdot s_2=s_2^{-1}s_1s_2s_1^{-1}=s_2^2s_1^2\cdot s_1^8.$$

However, the abelian group of order 16, type (2, 1, 1), does not admit such an isomorphism as s_1s_2 requires. The group does not exist.

CASE 2. $s_1^8=s_2^8=1$, $s_1s_2^2s_1$, $s_1^2s_2^2$, s_1s_2 and s_1^4 are all of order 2 and are all commutative, and hence generate an abelian group of order 16, type (1,1,1,1). This is invariant under s_1 and s_2.

$$s_1^{-1}s_1s_2^2s_1s_1=s_2^2s_1^2,\quad s_1^{-1}s_1^2s_2^2s_1=s_1s_2^2s_1,\quad s_1^{-1}s_1s_2s_1=s_2s_1=s_1s_2\cdot s_1s_2^2s_1,$$
$$s_2^{-1}s_1s_2^2s_1s_2=s_2^{-1}s_1s_2s_1^{-1}=s_2^{-2}s_1^{-2}=s_2^2s_1^2,\quad s_2^{-1}s_1^2s_2^2s_2=s_2s_1^2s_2=s_2s_1\cdot s_1s_2,$$
$$s_2^{-1}s_1s_2s_2=s_2^{-1}s_1s_2^2=s_2^{-1}s_1^{-1}s_1^2s_2^2=s_1s_2 . s_1^2s_2^2.$$

If this group of order 16 be extended by s_1, a group of order 64 results. The subgroup contains s_1s_2; hence G_{64} involves s_2. The existence of such a group is established by the substitutions $s_1=aebfcgdh$, $s_2=agbfcedh$. In the table this is No. 2 of order 64.

CASE 3. $s_1^4=s_2^4=1$. $s_1^2s_2^2$, s_1s_2 and $s_1s_2^2s_1$ are all of order 2 and all commutative. They generate an abelian group of order 8, type (1, 1, 1), which is invariant under s_1.

$$s_1^{-1}s_1^2s_2^2s_1=s_1s_2^2s_1,\quad s_1^{-1}s_1^2s_2^2s_1=s_1s_2^2s_1,\quad s_1^{-1}s_1s_2s_1=s_2s_1=s_1s_2\cdot s_1s_2^2s_1.$$

The order of G is therefore 32. Its existence is proved by the substitutions $s_1=abcd\cdot efgh$, $s_2=ag\cdot bfhd$. This group is No. 2 of order 32 in the table.

CASE 4. $s_1^2=s_2^2=1$, $(s_1s_2)^2=1$. The four-group results.

CASE 5. If $s_1s_2=s_2s_1$, one of three groups results. $s_1^8=s_2^8=1$ gives the abelian group of order 16, type (3, 1); $s_1^4=s_2^4=1$, the abelian group of order 8, type (2, 1); and $s_1^2=s_2^2=1$, the four-group.

The following theorem has thus been proved:

If two operators satisfy the three conditions $s_1^4=s_2^4$, $(s_1s_2)^2=1$, $s_1^2s_2^2=s_2^2s_1^2$, *they will generate one of two non-abelian or one of four abelian groups. The non-abelian groups are of orders* 64 *and* 32 *respectively, and the abelian groups are of orders* 16, 8, 4 *and* 2.

4. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^6=s_2^6$, $(s_1s_2)^2=1$, *when* s_1^3 *and* s_2^3 *Are Invariant.*

When s_1 and s_2 satisfy the above conditions, the theorem quoted in the preceding section may be applied to find the maximum order. s_1s_2 and s_2s_1 are of order 2; hence their product is transformed into its inverse by each operator.

$$(s_1s_2^2s_1)^2=s_1s_2^2s_1^2s_2^2s_1=s_1s_2^{-1}s_2^3s_1^3s_1^{-1}s_2^2s_1=s_1s_2^{-1}s_1^{-1}s_2^2s_1\cdot s_1^3s_2^3=s_1^2s_2^3s_1\cdot s_1^3s_2^3=s_1^6s_2^6=s_1^{12}.$$

The order of s_1 and s_2 then divides 24. If the common order of s_1 and s_2 is 2, G is the four-group. If this order is 3 or 4, G is the tetrahedral group or the abelian group of order 8 and type (2, 1) respectively. If $s_1^6=s_2^6=1$, the group generated is the first group of order 48 in the following list of substitution groups. $s_1^2s_2^2$, $s_2^2s_1^2$ are both of order 2 and generate the four-group which is invariant under s_1, so that $[s_1^2, s_1^2s_2^2]$ is the tetrahedral group.

$$(s_1^2s_2^2)^2=s_1^2s_2^2s_1^2s_2^2=s_1^2s_2^{-1}s_1^{-1}s_2^2s_1^3s_2^3=s_1^3s_2^3s_1^3s_2^3=1\,;$$

$$s_1^3s_1^2s_2^2s_1=s_1s_2^2s_1=s_1^2s_2^2\cdot s_2^2s_1^2,\quad s_1^4\cdot s_1^2s_2^2s_1^2=s_2^2s_1^2,\quad (s_1^2\cdot s_1^2s_2^2)^3=1.$$

This tetrahedral group is extended by s_1, giving a group of order 24 which is invariant under s_2; for $s_2^{-1}s_1s_2=s_2^4s_1^6$, $s_2^{-1}s_1^2s_2^2s_2=s_2^2s_1^2$. Hence, G is of order 48. It may be generated by the two operators $s_1=abd\cdot gh$, $s_2=abc\cdot ef$.

If $s_1^6=s_2^3=1$, it may be easily verified that the group is the non-twelve group of order 24, and may be generated by $s_1=ahdbgc\cdot ef$, $s_2=cehdfg$. If the order of both operators is 8, they are of course commutative and G is the abelian group of order 32 and type (3, 2). When $s_1^{12}=s_2^{12}=1$, G is of order 96. An invariant abelian subgroup of order 8 and type (2, 1) is generated by s_1^3 and s_2^3, and G is the direct product of this group and the tetrahedral group. It may be generated by the two substitutions $s_1=abc\cdot efgh\cdot ijkl$, $s_2=abd\cdot eigk\cdot fjhl$. The maximum group arises when $s_1^{24}=s_2^{24}=1$. It is easily shown that this group is of order 192 and is the direct product of the tetrahedral group and the abelian group of order 16 and type (3, 1). The following theorem results:

If two operators satisfy the conditions $s_1^6=s_2^6$, $(s_1s_2)^2=1$, $s_2^{-1}s_1^3s_2s_1^{-3}=s_1^{-1}s_2^3s_1s_2^{-3}=1$, *then they generate one of eight possible groups of orders* 192, 96, 48, 32, 24, 12, 8, 4, *respectively, three of which are abelian.*

5. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^8=s_2^4, \quad s_1^2s_2=s_2s_1^2, \quad (s_1s_2)^2=1$.

In order to find the upper limits for the orders of s_1 and s_2 when they satisfy the above relations, we proceed as before. Since s_1s_2 is of order 2, $s_1s_2^2s_1$ must be transformed into its inverse by s_1s_2 and s_2s_1.

$$(s_1s_2^2s_1)^4=s_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1=s_1^8s_2^8=s_1^{24}.$$

Since s_1^{24} is invariant under the group generated, the order of s_1 must divide 48. Five trivial cases arise when the orders of s_1 and s_2 are less than 8 and 4 respectively. If $s_1^2=s_2^2=1$, the group is the four-group, and if $s_1^4=s_2^2=1$, it is the octic group. $s_1^4=s_2^4=1$ define the group which is No. 5 of the list of substitution groups of order 16. If both operators are of order 3 or if one is of order 6 and the second of order 3, they generate the cyclic group of order 3 or the cyclic group of order 6, respectively.

If $s_1^6=s_2^4=1$, s_1 and s_2 generate an abelian subgroup of order 16, type (2, 2), which is invariant under s_1, for $s_1^{-1}s_2s_1=(s_2s_1^2)^{-1}$. G is then of order 32 and is the sixth of that order in the list of substitution groups. It may be generated by the two substitutions $s_2=abcd\cdot ef\cdot gh$, $s_1=aebfcgdh$.

When the common order of s_1 and s_2 is 12, it is readily seen that s_1^3 and s_2^3 satisfy the conditions for the group of order 16 mentioned in the first paragraph of this section and that G is the direct product of this group of order 16 and a cyclic group of order 3.

$s_1^{24}=s_2^{12}=1$ likewise gives rise to a group which is a direct product. s_1^3 and s_2^3 generate the above group of order 32, and G is the direct product of this and the cyclic group of order 3. It is of order 96. The only condition remaining to be considered is $s_1^{48}=s_2^{24}=1$. That these conditions are contradictory follows, if we consider the transform $(s_1s_2)^{-1}s_2s_1s_1s_2=s_1s_2^2s_1^2s_2=s_1^3s_2^3$. $s_1^3s_2^3$ should be of order 2, while this assumption leads to a contradiction.

$$(s_1^3s_2^3)^2=(s_2^{-3}s_1^{-3})^2=(s_2^{-4}s_1^{-8}s_2s_1^5)^2=(s_1^{-12}\cdot s_2s_1)^2=s_1^{24}\neq 1.$$

If two operators s_1 *and* s_2 *satisfy the conditions* $s_1^8=s_2^4$, $s_1s_2s_1^{-2}s_2^{-1}=(s_1s_2)^2=1$, *they generate one of three abelian groups or five non-abelian groups. The non-abelian groups are of orders* 96, 48, 32, 16 *and* 8, *and the abelian groups are the cyclic groups of orders* 6 *and* 3 *and the four-group.*

6. *The Groups Generated by Two Operators Satisfying the Two Conditions* $s_1^4=s_2^4, \quad (s_1s_2)^3=(s_1^2s_2)^2=1$.

In order to find the upper limit for the orders of s_1 and s_2, consider the product $s_1^2s_2\cdot s_2s_1^2$. $s_1^2s_1$ is of order 2; hence, the product $s_1^2s_2\cdot s_2s_1^2$ is transformed into its inverse by each operation. $(s_2s_1^2\cdot s_1^2s_2)^2=(s_2^2s_1^4)^2=s_2^{12}$. s_2^{12} is invariant and hence the order of s_1 and s_2 must divide 24. $s_1^2=s_2^2=1$ define the symmetric group of order 6; $s_1^3=s_2^3=1$ define the cyclic group of order 3. That $s_1^4=s_2^4=1$ define the octahedral group follows if the operators s_1 and s_1s_2 are considered. $s_1^4=(s_1s_2)^3=(s_1^2s_2^2)^2=1$. Thus they fulfil the well-known relations of the octahedron group. The group of order 18 which is represented as a substitution group by (abc) all (def) is defined abstractly by the given relations when $s_1^6=s_2^6=1$.

When $s_1^8=s_2^8=1$, s_1^2 and s_2^2 generate a quaternion subgroup which is invariant under s_1s_2.

$$(s_1^2)^4=(s_2^2)^4=s_1^4s_2^4=1, \quad s_1^6s_2^2s_1^2=s_2^6, \quad (s_1s_2)^{-1}s_1^2s_1s_2=s_2^7s_1^2s_2=s_1^2s_2s_1^2\cdot s_1^2s_2=s_1^2s_2^6,$$
$$(s_1s_2)^{-1}s_2^2s_1s_2=s_2^7s_1^7s_2^2s_1s_2=s_1^2s_2s_1s_2^2s_1s_2=s_1s_2^7s_1^7s_2s_1s_2=s_1s_2^7s_1^6s_2^7s_1^7=s_1^6s_2^8s_1^7=s_1^2.$$

$[s_1^2,\ s_2^2,\ s_1s_2]$ is evidently the non-twelve group of order 24. s_2^2 is in this subgroup, and the subgroup is invariant under s_2. Hence, G is of order 48.

$$s_2^{-1}\cdot s_1s_2\cdot s_2=s_2^7s_1s_2^2=s_2s_1^3=s_2s_1\cdot s_1^2, \quad s_2^{-1}s_1^2s_2=s_1^2s_2s_1^2\cdot s_1^2s_2=s_1^2s_2^6.$$

This group may be generated by $s_1=acegdfbh$, $s_2=aehfdbgc$. It is the fourth group of order 48 in the list which follows.

That the two orders $s_1^{12}=s_2^{12}=1$ and $s_1^{24}=s_2^{24}=1$ are contradictory is seen as follows:

$$s_1^{-2}s_2s_1^2=s_2^{-1} \text{ and } s_1^{-2}s_2^4s_1^2=s_2^{-4}=s_2^4, \ \therefore\ s_2^8=1.$$

If two operators satisfy the conditions $s_1^4=s_2^4$, $(s_1s_2)^3=(s_1^2s_2)^2$, *they generate one of four non-abelian groups or one abelian. The orders of the non-abelian groups are* 48, 24, 8 *and* 6; *and of the abelian*, 3.

7. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^6=s_2^6,\quad (s_1s_2)^2=1,\quad s_2^{-1}s_1^2s_2=s_1^2.$

In order to find the maximum order of s_1 and s_2, the same process is used as in the preceding problems. Since s_1s_2 and s_2s_1 are of order 2, their product must be transformed into its inverse by each operator.

$$(s_1s_2\cdot s_2s_1)^3=s_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1=s_1s_2^6s_1^6=s_1^{12}.$$

The order of s_1 and s_2 must then divide 24, since s_1^{12} is invariant and is transformed into its inverse.

The conditions are satisfied if $s_1^2=s_2^2=1$, in which case G is the four-group, if $s_1^2=s_2^3=1$, where G is the dihedral group of order 6, when G is cyclic of order 3 and when $s_1^4=s_2^4=1$, which gives the abelian group of order 8 and type $(2,1)$. The dihedral group of order 12 arises when $s_1^2=s_2^6=1$. When $s_1^6=s_2^3=1$, $[s_1^2,\ s_2]$ is the non-cyclic group of order 9 and is invariant under s_1. Hence, G is of order 18. It may be generated by $s_1=acdfgi\cdot bhe$, $s_2=abc\cdot def\cdot ghi$.

When $s_1^6=s_2^6=1$, $[s_1^2,\ s_1s_2^2s_1]$ is an invariant non-cyclic subgroup of order 9. $s_1s_2\cdot s_1s_2^2s_1\cdot s_1s_2=s_2s_1^2s_2=(s_1s_2^2s_1)^{-1}$. Hence, $[s_1^2,\ s_1s_2^2s_1,\ s_1s_2]$ is of order 18. It is invariant under s_1, giving 36 as the order of G. This is the second group of order 36 in the list of substitution groups, and may be generated by the substitutions $s_1=adcbfe$, $s_2=abc\cdot gh$.

If $s_1^8=s_2^8=1$, $[s_1^2,\ s_1s_2]$ gives the abelian group of order 8 and type (2, 1) which is invariant under s_2, $s_2^{-1}s_1s_2^2=s_2^{-1}s_1^3=(s_1^5s_2)^{-1}$. G is the eighth group of order 16 in the list of substitution groups. When $s_1^{12}=s_2^{12}=1$, $[s_2,\ s_1^2]$ is an invariant abelian subgroup of order 36 with the invariants 12 and 3. G is of order 72. It is generated by

$$s_1=agce\cdot bhdf\cdot imlpoj\cdot kqn,\quad s_2=abcd\cdot efgh\cdot ijk\cdot lmn\cdot opq.$$

When $s_1^4=s_2^{24}=1$, G is the group of order 24 which contains a cyclic subgroup of order 12, each operator of which is transformed into its fifth power by an operator of order 2. $s_1^{-1}s_2s_1=s_1^{-2}s_2^{-1}=s_2^5$. When $s_1^8=s_2^{24}=1$, G is of order 48 and is the direct product of the symmetric group of order 6 and the cyclic group of order 8.

If $s_1^{24}=s_2^{24}=1$, $[s_2,\ s_1^2]$ is abelian of order 72, with invariants 24 and 3. It is invariant under s_1. Hence, G is of order 144. As generating substitutions may be taken $s_1=ahcbedgf\cdot imlpoj\cdot kqn$, $s_2=abcdefgh\cdot ijk\cdot lmn\cdot opq$.

If two operators satisfy the conditions $s_1^6=s_2^6$, $(s_1s_2)^2=1$, $s_2^{-1}s_1^2s_2=s_1^2$, *they generate one of nine non-abelian groups of orders* 144, 72, 48, 36, 24, 18, 16, 12 *and* 6 *respectively, or one of four abelian groups of orders* 8, 6, 4 *or* 3.

8. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^6=s_2^6,\quad (s_1s_2)^3=1,\quad s_1^2s_2^2=s_2^2s_1^2,\quad (s_1^3s_2^3)^2=1.$

It is possible to apply again the theorom stated, that if two operators have a common square, the product of the one into the inverse of the other is transformed into its inverse by each operator. s_1^3 and s_2^3 have a common square.

$$(s_1^3s_2^{-3})^2=s_1^3s_2^{-3}s_1^3s_2^{-3}=s_1^3s_2^3s_1^3s_2^3s_2^{-12}=s_2^{-12}.$$

Since s_2^{-12} both is invariant and is transformed into its inverse, the orders of s_1 and s_2 divide 24.

If now s_2s_1 be transformed by $s_1^3s_2^3$, the resulting operator must be of order 3.

$$s_1^3s_2^3\cdot s_2s_1\cdot s_1^3s_2^3=s_1^3s_2^4s_1^4s_2^3=s_1s_2\cdot s_1^{12}.$$

From this it is readily seen that the order of s_1 must divide 36. Twelve is then the largest possible order for s_1 and s_2.

$s_1^3=s_2^3=1$ gives the non-cyclic group of order 9. $s_1^2=s_2^3=1$ generates the tetrahedral group. When $s_1^2=s_2^6=1$, G may be generated by $s_1=ab$, $s_2=aecbfd$, and is of order 24. When $s_1^3=s_2^6=1$, s_2^3 and $s_1^{-1}s_2^3s_1$ generate the four-group, since

$$s_1^{-1}s_2^3s_1s_2^3\cdot s_1^{-1}s_2^3s_1s_2^3=s_1^2s_2s_1s_2s_1^2s_2s_1s_2^3=s_1s_2^{-1}s_2^{-1}s_1^2s_2^2=s_1^3=1.$$

s_1 with this generates the tetrahedral group which is invariant under s_2.

$$s_2^{-1}s_1^{-1}s_2^3s_1s_2=s_1s_2s_1s_2^4s_1s_2=s_1s_2s_1^2s_2^5=s_1s_2^3s_1^2s_2^3, \quad s_2^{-1}s_1s_2=s_2^3s_1s_2^3.$$

G is of order 36 and may be generated by the two substitutions

$$s_1=abg\cdot cdi\cdot efk\cdot hlj, \quad s_2=abcdef\cdot ghijkl.$$

When $s_1^4=s_2^{12}=1$, s_1 and s_2 generate the dicyclic group of order 24.

$$s_1^{-1}s_2s_1=s_1^3s_2s_1=s_2s_1s_2^{11}s_1=s_2s_1s_2^2s_1\cdot s_2^9=s_2s_1^2s_2s_1^2\cdot s_2^9=s_2^2s_2^9=s_2^{11}.$$

When $s_1^6=s_2^6=1$, s_1^3, s_2^3, $s_1^{-1}s_2^3s_1$ and $s_2^{-1}s_1^3s_2$ generate an abelian subgroup of order 16 and type $(1, 1, 1, 1)$.

$$(s_1^3\cdot s_1^5s_2^3s_1)^2=s_1^2s_2^3s_1s_2^2s_2^3s_1=s_1^2s_2^3s_1^3s_2^3s_1=s_1^2s_2^6s_1^4=1,$$
$$(s_2^3\cdot s_1^5s_2^3s_1)^2=s_2^3s_1^5s_2^3s_1s_2^3s_1^5s_2^3s_1=s_2^3s_1^2s_2^3s_1s_2^3s_1^5s_2^3s_1^4=s_2^5s_1^2s_2s_1s_2s_1^2s_2^5s_1=s_2^5s_1s_2^5s_1s_2^5s_1=1,$$
$$(s_1^3\cdot s_2^5s_1^2s_2)^2=s_2^3s_1^3s_2^2s_1^3s_2s_1^3s_2^5s_1^3s_2=s_2^3s_1s_2^2s_1^5s_2s_1s_2^2s_1^5s_2^4=s_2^3\cdot(s_1^5s_2)^3s_2^3=1,$$
$$(s_2^3\cdot s_2^5s_1^3s_2)^2=s_2^2s_1^3s_2^3s_1^3s_2=s_2^5s_1^6s_2=s_2^6=1.$$

This abelian subgroup is invariant under s_1 and

$$s_2\cdot s_1^{-1}\cdot s_1^5s_2^3s_1\cdot s_1^2=s_1^5s_2^3s_1s_2^3, \quad s_2^{-1}s_1^5s_2^3s_1\cdot s_2=s_2^5s_1^3s_2^3s_1s_2=s_1^3s_2s_1^5s_2s_1s_2^3s_1^4s_2=s_1^5s_2s_1^4s_2^2s_1^3=s_1^5s_2^3s_1;$$
$$s_1^{-1}\cdot s_2^5s_1^3s_2\cdot s_2=s_2^4\cdot s_1^3s_2^2=s_2^4\cdot s_2^4s_1^3s_2s_1^3=s_2^5s_1^3s_2s_1^3.$$

The group is then generated by this subgroup and s_1, s_2, and is of order 144. It is generated by the two substitutions $s_1=abc\cdot ef\cdot gh$, $s_2=ab\cdot cd\cdot efg$. In the list of substitution groups which follows, this is the first group of order 144.

When the common order of s_1 and s_2 is 12, it is possible again to consider the subgroup generated by the four operators s_1^3, s_2^3, $s_1^{-1}s_2^3s_1$, $s_2^{-1}s_1^3s_2$. These now generate the group of order 32.

$$s_1^{-1}s_2^9s_1\cdot s_1^3s_1^{-1}s_2^3s_1=s_1^{-1}s_2^9s_1^3s_2^3s_1=s_1^3, \quad s_1^{-1}s_2^9s_1\cdot s_2^3s_1^{-1}s_2^3s_1=s_1^{-1}s_2^3s_1s_2^3s_1^{-1}s_2s_1=s_2^9,$$
$$s_2^{-1}s_1^9s_2s_1^3s_2^{-1}s_1^3s_2=s_2^5s_1^3s_2s_1^3s_2^5s_1^3s_2=s_1^9, \quad s_2^{-1}s_1^9s_2s_2^3s_2^{-1}s_1^3s_2=s_2^3,$$
$$s_1^{-1}s_2^9s_1s_2^{-1}s_1^3s_2s_1^{-1}s_2^3s_1=s_2^{-1}s_1^3s_2.$$

This G_{32} is invariant under s_1 and s_2, and G is of order 288. In this G_{288}, s_1^6 is an invariant operator of order 2. With respect to this invariant operator of order 2, the quotient group is the G_{144} considered above. This G_{144} contains an invariant abelian subgroup of order 16 and type $(1, 1, 1, 1)$. Corresponding to this in G_{288} must be an invariant subgroup of order 32, all of whose operators must be of order 2 or 4. The invariant subgroup in the quotient group is transformed according to a non-cyclic group of order 9, in the group of isomorphisms of the quotient group. Hence, the corresponding sets in G_{288} must be generated in the same way. We find on trial that the group of order 32 which admits this isomorphism is the group of order 32 which involves nineteen operators of order 2 and twelve of order 4, of which the latter have a common square. That such a group of order 288 exists, is established by the two substitutions

$$s_1 = actnp\varepsilon bdsmo\theta \cdot e\alpha yixvf\beta zjwu \cdot glhk \cdot q\gamma r\delta,$$
$$s_2 = ac\alpha pnvbd\beta omn \cdot elfk \cdot gtri\gamma\varepsilon hsqj\delta\theta \cdot wzxy.$$

The above results may be stated in the following theorem:

If two operators s_1 *and* s_2 *satisfy the conditions* $s_1^6 = s_2^6$, $(s_1s_2)^3 = (s_1^3s_2^3)^2 = 1$, $s_1^2s_2^2 = s_2^2s_1^2$, *they must generate one of seven groups of order* 9, 12, 24, 36, 144 *or* 288.

II. TABLE OF ABSTRACT DEFINITIONS OF THE SUBSTITUTION GROUPS OF DEGREE 8.

The substitution groups used in this table are taken from the list of Professor G. A. Miller (AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 326). The notation there used has been practically retained. There are, in all, 200 distinct substitution groups of degree 8. As abstract groups, however, these are not all distinct. The same abstract group may be represented in more than one way on eight letters. Some of the groups of degree 8 may be abstractly the same as substitution groups of lower degree. The abstract definitions of the substitution groups whose degree does not exceed 7 have already been given.* The definitions which are given in the present table exclude the groups which may be represented on a smaller number of letters, and include only the distinct abstract groups of degree 8. There are 82 such groups to be defined.

	Order.	No.	Designation.	Abstract Definition.
1	8	1	$(abcdefgh)$	$s_1^8 = 1.$
2		2	$(abcdefgh)_8$	$s_1^4 = (s_1s_2^{-1})^4 = s_1^2s_2^2 = 1.$
3	15	1	$(abcde)\,(fgh)$	$s_1^{15} = 1.$
4	16	1	$(ab)\,(cd)\,(ef)\,(gh)$	$(s_is_j)^2 = 1 \ (i, j = 1, 2, 3, 4).$
5		2	$(abcd)\,(ef)\,(gh)$	$s_1^4 = s_2^2 = s_3^2 = s_i^{-1}s_j^{-1}s_is_j = 1 \ (i, j = 1, 2, 3).$
6		3	$(abcd)\,(efgh)$	$s_1^4 = s_2^4 = s_1^3s_2^3s_1s_2 = 1.$
7		4	$[(abcd)\,(efgh)_8]\,\dim_1$	$s_1^4 = s_2^4 = s_1^3s_2s_1s_2 = 1.$
8		5	$[(abcd)\,(efgh)_8]\,\dim_1$	$s_1^4 = s_2^4 = (s_1s_2)^2 = (s_1^2s_2^2)^2 = 1.$
9		6	$[(abcd)\,(efgh)_8]\,\dim_2$	$s_1^4 = s_2^2 = s_3^2 = (s_1s_2)^2 = (s_2s_3)^2 = s_1^3s_2s_1s_2 = 1.$

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* G. A. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), p. 383.

	Order.	No.	Designation.	Abstract Definition.
10		7	$[(abcd)(efgh)]\,\mathrm{pos}\,(ae\cdot bf\cdot cg\cdot dh)$	$s_1^4=s_2^2=s_1^3s_2s_1s_2=s_3^2=s_3s_1^3s_3s_1=s_3s_2s_3s_1^2s_2=1.$
11		8	$[(abcd)(efgh)]\,\mathrm{pos}\,(aefbgchd)$	$s_1^8=s_2^2=s_2s_1s_2s_1^3=1.$
12		9	$(abcd\cdot efgh)_{8,2}(aebfcgdh)$	$s_1^8=s_2^2=s_2s_1s_2s_1^5=1.$
13		10	$(abcd)_8(efgh)_8(afbgchde)$	$s_1^8=s_2^2=(s_1s_2)^2=1.$
14	18	1	$(abc)(def)(gh)$	$s_1^6=s_2^3=s_2^2s_1^5s_2s_1=1.$
15	24	1	$(abc\cdot def)(afdc\cdot egbh)$	$s_1^3=s_2^6=s_1s_2^4s_1s_2=1.$
16	30	1	$(abcde)(fgh)\,\mathrm{all}$	$s_1^{15}=s_2^2=s_2s_1s_2s_1^4=1.$
17		2	$(abcde)_{10}(fgh)$	$s_1^{15}=s_2^2=s_2s_1s_2s_1^{11}=1.$
18		3	$[(abcde)_{10}(fgh)\,\mathrm{all}]\,\mathrm{dim}$	$s_1^2=s_2^2=(s_1s_2)^{15}=1.$
19	32	1	$(abcd)_8(ef)(gh)$	$s_1^4=s_2^2=s_3^2=s_4^2=(s_1s_2)^2=s_3s_1^3s_3s_1=(s_3s_2)^2=s_4s_1^3s_4s_1=(s_4s_3)^2=(s_4s_2)^2=1.$
20		2	$(abcd)_8(efgh)$	$s_1^4=s_2^4=(s_1s_2)^2=(s_1^2s_2^2)^2=1.$
21		3	$[(abcd)_8(efgh)_8]\,\mathrm{dim}$	$s_1^4=s_2^4=s_3^2=s_1^3s_2^3s_1s_2=(s_2s_3)^2=(s_1s_3)^2=1.$
22		4	$[(abcd)_8(efgh)_8]\,\mathrm{dim}$	$s_1^4=s_2^4=s_3^2=(s_1s_2)^2=(s_1s_3)^2=s_3s_2s_3s_2^3=s_1^3s_2^2s_1s_2^2=1.$
23		5	$[(abcd)_8(efgh)_8]\,\mathrm{dim}$	$s_1^4=s_2^4=s_3^2=(s_1s_2)^2=s_1^3s_2^2s_1s_2^2=(s_1s_3)^2=s_3s_2s_3s_1^2s_2^3=1.$
24		6	$(abcd)(efgh)(ae\cdot bf\cdot cg\cdot dh)$	$s_1^8=s_2^4=(s_1s_2)^2=s_1^6s_2^3s_1^2s_2=1.$
25		7	$[(abcd)_8(efgh)_8]_{2,2}\,(ae\cdot bf\cdot cg\cdot dh)$	$s_1^4=s_2^4=s_3^2=s_4^2=s_1^3s_2s_1s_2=s_3s_1s_3s_1^3=s_2^3s_1s_2s_1=(s_4s_1)^2=(s_4s_2)^2=(s_4s_3)^2=(s_3s_2)^2=1.$
26		8	$[(abcd)_8(efgh)_8]_{2,2}(aebf\cdot cgdh)$	$s_1^4=s_2^2=(s_1s_2)^4=s_1^2s_2s_1s_2s_1^2s_2s_1^3s_2=1.$
27		9	$[(abcd)_8(efgh)_8]\,(aebfcgdh)$	$s_1^8=(s_1s_2)^2=s_1^4s_2^4=s_2^2s_1^{-1}s_2^2s_1=1.$
28		10	$[(abcd)_8(efgh)_8]_{2,2}\,(ae\cdot bf\cdot cg\cdot dh)$	$s_1^8=s_2^2=s_3^2=s_2s_1s_2s_1^7=s_3s_1s_3s_1^5=(s_2s_3)^2=1.$
29	36	1	$[(abc)\,\mathrm{all}\,(def)\,\mathrm{all}]\,\mathrm{pos}\,gh$	$s_1^6=s_2^6=s_2^5s_1^5s_2s_1=s_3^2=(s_1s_3)^2=(s_2s_3)^2=s_1^3s_2^3=1.$
30		2	$(abcdef)_{18}(gh)$	$s_1^6=s_2^6=(s_1s_2)^2=s_1^4s_2^5s_1^2s_2=1.$
31	48	1	$(abcd)\,\mathrm{pos}\,(ef)(gh)$	$s_1^6=s_2^6=(s_1s_2)^2=s_2^3s_1^5s_2^3s_1=s_1^3s_2^5s_1^3s_2=1.$
32		2	$(abcd)\,\mathrm{pos}\,(efgh)$	$s_1^{12}=s_2^{12}=s_1^6s_2^9=(s_1s_2)^2=1.$
33		3	$[(abcd)\,\mathrm{pos}\,(efgh)\,\mathrm{pos}]_{4,4}$	$s_1^3=s_2^3=s_3^3=(s_1s_2)^2=(s_1s_3)^2=(s_2^2s_3)^2=(s_3s_2s_1^2)^2=1.$
34		4	$(abc\cdot def)(aefgdbch)$	$s_1^8=s_2^8=(s_1s_2)^3=s_1^4s_2^4=s_2^5s_1^6s_2s_1^2=1.$

	Order.	No.	Designation.	Abstract Definition.
35	56	1	$(abcdefgh)_8(bcedghf)$	$s_1^7=s_2^7=(s_1s_2)^2=s_1s_2^3s_1^4s_2^2=1.$
36	60	1	$(abcde)_{10}(fgh)$ all	$s_1^6=s_2^2=(s_1s_2)^{10}=(s_1s_2^2)^2=1.$
37		2	$(abcde)_{20}(fgh)$	$s_1^{15}=s_2^4=s_2^3s_1s_2s_1^2=1.$
38		3	$[(abcde)_{20}(fgh)$ all$]$ pos	$s_1^{15}=s_2^4=s_2^3s_1s_2s_1^7=1.$
39	64	1	$(abcd)_8(efgh)_8$	$s_1^2=s_2^2=s_3^2=s_4^2=(s_1s_2)^4=(s_1s_3)^2$ $=(s_1s_4)^2=(s_3s_4)^2=(s_2s_3)^2$ $=(s_2s_4)^2=1.$
40		2	$[(abcd)_8(efgh)_8]$ dim $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^8=s_1^4s_2^4=(s_1s_2)^2=s_1^7s_2^2s_1s_2^2$ $=s_3^2=s_3s_1s_3s_1^5=(s_3s_2)^2=1.$
41		3	$[(abcd)_8(efgh)_8]$ dim $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^4=s_2^2=s_3^2=(s_1s_2)^4=(s_1s_3)^2$ $=(s_2s_3)^2=s_1^2s_2s_1s_2s_1^2s_2^2s_1^3s_2=1.$
42		4	$(abcd)_8(efgh)_8$ dim $(aecg\cdot bf\cdot dh)$	$s_1^4=s_2^4=(s_1s_2)^2=(s_1s_2^2)^4$ $=(s_1^3s_2)^4=1.$
43		5	$[(abcd)_8(efgh)_8$ pos$]$ dim $(aebfcgdh)$	$s_1^8=s_2^4=(s_1s_2)^2=(s_1^2s_2^2)^2=1.$
44	72	1	(abc) all (def) all (gh)	$s_1^6=s_2^6=s_3^2=(s_1s_2)^3=s_1^5s_2^2s_1s_2^2$ $=s_2^5s_1^2s_2s_1^2=s_3s_1s_3s_1^5=s_3s_2s_3s_2^5=1.$
45		2	$(abcdef)_{36}(gh)$	$s_1^4=s_2^4=s_3^2=(s_1s_2)^2=(s_1^2s_2^2)^3$ $=(s_1s_2^3)^3=s_3s_1s_3s_1^3=s_3s_2s_3s_2^2=1.$
46	96	1	$(abcd)$ all (ef) (gh)	$s_1^2=s_2^6=s_3^2=(s_1s_2^2)^4=(s_1s_2^5)^2$ $=(s_1s_3)^2=s_3s_2s_3s_2^5=1.$
47		2	$(abcd)$ all $(efgh)$	$s_1^{12}=s_2^{12}=s_3^2=s_1^3s_2^9=(s_1s_2)^2$ $=s_3s_1s_3s_1^7=s_3s_2s_3s_2^7=1$
48		3	$(abcd)$ pos $(efgh)_8$	$s_1^{12}=s_2^{12}=s_3^2=s_1^3s_2^9=(s_1s_2)^2$ $=s_3s_1s_3s_1^5=s_3s_2s_3s_2^5=1.$
49		4	$[(abcd)$ all $(efgh)_8$ com$]$ dim	$s_1^6=s_2^4=(s_1s_2^{-1})^4=(s_1s_2)^2$ $=(s_1^3s_2^2)^2=1.$
50		5	$[(abcd)$ all $(efgh)_8$ cyc$]$ dim	$s_1^{12}=s_2^4=(s_1^6s_2^2)^2=(s_1^5s_2)^2$ $=(s_1s_2^2)^3s_1^9=1.$
51		6	$[(abcd)$ all $(efgh)$ all$]_{4,4}$	$s_1^4=s_2^4=(s_1s_2)^3=(s_1^2s_2^2)^2$ $=(s_1^2s_2)^4=1.$
52		7	$[(abcd)$ pos $(efgh)$ pos$]_{4,4}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^6=s_2^6=(s_1s_2)^2=(s_1^3s_2^3)^2$ $=(s_1s_2^3)^3=1.$
53		8	$[(ab)(cd)(ef)(gh)]$ pos $(ach\cdot bdg)(ade\cdot bcf)$	$s_1^6=s_2^2s_1^3=s_2^2s_3^3=s_1^2s_3s_1^2s_3^4$ $=s_1s_2s_1^2s_3s_2s_3^5=1.$
54	120	1	$(abcde)_{20}(fgh)$ all	$s_1^{15}=s_2^4=s_3^2=s_2^3s_1s_2s_1^7=s_3s_2s_3s_2^3$ $=s_3s_1s_3s_1^4=1.$

	Order.	No.	Designation.	Abstract Definition.
55	128	1	$(abcd)_8(efgh)_8(ae\cdot bf\cdot cg\cdot dh)$	$s_1^8=s_1^4s_2^4=(s_1s_2)^2=(s_1^3s_2)^2=s_3^2=s_4^2$ $=s_1^7s_2^2s_1s_2^2=s_3s_1s_3s_1^5=(s_3s_2)^2$ $=s_4s_1s_4s_2^7=(s_4s_3)^2s_1^3s_2^7=1.$
56	144	1	$(abcd)\text{pos}(efgh)$	$s_1^6=s_2^6=(s_1s_2)^3=(s_1^3s_2^3)^2$ $=s_1^2s_2^2s_1^4s_2^4=1.$
57		2	$(abcdef)_{72}(gh)$	$s_1^2=s_2^4=s_3^2=(s_1s_2)^6=(s_1s_2^3s_1s_2)^2$ $=(s_1s_3)^2=s_3s_2s_3s_2^3=1.$
58	168	1	$(abcdefgh)_8(bcedghf)_{21}$	$s_1^6=s_2^6=(s_1s_2)^7=(s_1^3s_2^3)^2=(s_1s_2^3)^3$ $=s_2^2s_1^3s_2s_1^5s_2^3s_1^4=s_1^5s_2^3s_1s_2^4s_1^3s_2^5=1.$
59	180	1	$(abcde)\text{pos}(fgh)$	$s_1^{15}=(s_1^6s_2^6)^2=(s_2^5s_1^{10})$ $=(s_2^6s_1^6s_2^3)^3=1.$
60	192	1	$(abcd)\text{all}(efgh)^8$	$s_1^6=s_2^4=(s_1s_2)^2=(s_1s_2^{-1})^4$ $=(s_1^3s_2^2)^2=s_3^2=(s_3s_1)^2$ $=s_3s_2s_3s_1^3s_2s_1^3=1.$
61		2	$[(abcd)\text{all}(efgh)\text{all}]_{4,4}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^6=s_2^4=(s_2s_1^3)^2=(s_1^2s_2^2)^3$ $=(s_1s_2)^4=1.$
62		3	$[(ab)(cd)(ef)(gh)]$ $(ace\cdot bdf)(ceg\cdot dfh)$	$s_1^6=s_2^6=(s_1s_2)^4=s_1^3s_2s_1^3s_2^5=1.$
63		4	$+[(ab)(cd)(ef)(gh)]\text{pos}$ $(ace\cdot bdf)(eg\cdot fh)$	$s_1^4=s_2^6=(s_1s_2)^4=s_1^3s_2^3s_1s_2$ $=(s_1^2s_2^2)^3s_2^3=s_1^3s_2^6s_1^2s_2s_1s_2s_1^2s_2^2=1.$
64		5	$\pm[(ab)(cd)(ef)(gh)]$ $(ace\cdot bdf)(agbh)$	$s_1^8=s_2^8=(s_1^2s_2^6)^2=(s_1s_2)^3$ $=(s_1s_2^{-1})^3=(s_1s_2^2)^4=(s_1^2s_2)^4=1.$
65	288	1	$(abcd)\text{all}(efgh)\text{pos}$	$s_1^{12}=s_2^{12}=(s_1s_2)^6=(s_2s_1)^2s_2^9s_1^9$ $=(s_1^4s_2s_1s_2)^2=(s_1^2s_2^{-1})^2$ $=s_2^3s_1^4s_2^9s_1^8=s_1^3s_2^4s_1^9s_2^8=1.$
66		2	$[(abcd)\text{all}(efgh)\text{all}]\text{pos}$	$s_1^6=s_2^6=s_3^2=(s_1s_2)^3=s_1^2s_2^2s_1^4s_2^4$ $=(s_1^3s_2^3)^2=(s_1s_3)^2=(s_2s_3)^2=1.$
67		3	$(abcd)\text{pos}(efgh)\text{pos}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^6=s_2^2=(s_1s_2)^6=(s_1^2s_2)^2(s_2s_1^2)^{-2}$ $=(s_1^3s_2)^4=1.$
68	336	1	$(abcdefgh)_{336}$	$s_1^7=s_2^8=(s_1^5s_2)^6=(s_1s_2)^2$ $=(s_2s_1^4s_2s_1^3)^2=(s_2s_2^4s_2)^4=1.$
69	360	1	$(abcde)\text{all}(fgh)$	$s_1^{12}=s_2^5=(s_1^6s_2)^3=(s_1^9s_2^3)^2$ $=s_1^5s_2s_1^7s_2^4=1.$
70		2	$(abcde)\text{pos}(fgh)\text{all}$	$s_1^{15}=s_2^5s_1^{10}=(s_1^6s_2^6)^2=(s_2^6s_1^6s_2^3)^3$ $=s_3^2=(s_3s_2)^2=(s_3s_1)^2=1.$
71		3	$[(abcde)\text{all}(fgh)\text{all}]\text{pos}$	$s_1^{15}=s_2^5s_1^{10}=s_3^2=(s_2^6s_1^6)^2=(s_2^6s_1^6s_2^3)^3$ $=s_3s_1s_3s_2^4=s_3s_2s_3s_1^4=1.$
72	384	1	$(ab)(cd)(ef)(gh)(ace\cdot bdf)$ $(cg\cdot dh)$	$s_1^4=s_2^6=(s_1s_2)^5=(s_2s_1s_2^3s_1)^3$ $=(s_1s_2^{-1})^4=(s_1^2s_2s_1s_2)^2=1.$

	Order.	No.	Designation.	Abstract Definition.
73	576	1	$(abcd)\text{all}(efgh)\text{all}$	$s_1^{12}=s_2^{12}=s_1^4s_2^4s_1^8s_2^8=s_1^3s_2^3s_1^9s_2^9$ $=(s_1^3s_2^4)^2=(s_2^3s_1^4)^2=1.$
74		2	$[(abcd)\text{all}(efgh)\text{all}]\text{pos}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^6=s_2^2=s_3^2=(s_1s_2)^6=(s_1^3s_2)^4$ $=(s_1^2s_2)^2(s_2s_1^2)^{-2}=(s_3s_1)^2$ $=(s_3s_2)^2=1.$
75		3	$[(abcd)\text{all}(efgh)\text{all}]\text{pos}$ $(aebf\cdot cg\cdot dh)$	$s_1^6=s_2^4=(s_1s_2^2)^2=(s_1^3s_2^3)^4$ $=(s_2^3s_1s_2s_1)^3=1.$
76	720	1	$(abcde)\text{all}(fgh)\text{all}$	$s_1^{12}=s_2^{10}=(s_1^6s_2^2)^3=(s_1^9s_2^6)^2$ $=(s_1s_2)^2=s_1^{11}s_2^5s_1^5s_2^5$ $=s_2^9s_1^4s_2s_1^4=s_1^4s_2^2s_1^8s_2^{10}=1.$
77		2	$(abcdef)\text{pos}(gh)$	$s_1^8=s_2^4=s_3^2=(s_1s_2^2)^4=(s_1s_2^3s_1^2s_2)^2$ $=s_3s_1s_3s_1^2=s_3s_2s_3s_2^3=1.$
78	1152	1	$(abcd)\text{all}(efgh)\text{all}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^{12}=s_2^2=(s_1^4s_2)^2(s_1^8s_2)^2$ $=(s_1^3s_2)^2(s_1^9s_2)^2=(s_1^3s_2s_1^4s_2)^2=1.$
79	1344	1	$(abcdefgh)_{1344}$	$s_1^7=s_2^7=s_3^3=(s_1s_2)^2=(s_1^2s_2^2)^2$ $=(s_1^3s_2^3)^2=s_1s_2^3s_1^4s_2^2=(s_1s_3)^2$ $=(s_3s_1^5)^4=(s_3s_2^{-1})^2=(s_3s_2^2)^4=1.$
80	1440	1	$(abcdef)\text{all}(gh)$	$s_1^6=s_2^4=(s_1^2s_2^2)^4=(s_1^2s_2^3s_1^4s_2)^2$ $=s_1^5s_2^3s_1(s_1^2s_2)^2s_2(s_1^2s_2)^3s_2$ $=s_3^2=s_3s_1s_3s_1^5=s_3s_2s_3s_2^3=1.$
81	20160	1	$(abcdefgh)\text{pos}$	*$s_1^3=s_i^2=1\ (i=1, 2, \ldots, 7).$ $(s_is_{i+1})^3=1\ (i=1, 2, \ldots, 6).$ $(s_is_j)^2=1\ (i=1, 2, \ldots, 5)$ $(j>i+1).$
82	40320	1	$(abcdefgh)\text{all}$	*$s_i^2=1\ (i=1, 2, \ldots, 8).$ $(s_is_{i+1})^3=1\ (i=1, 2, \ldots, 7).$ $(s_is_j)^2=1\ (i=1, 2, \ldots, 6)$ $(j>i+1).$

III. EXPLANATORY NOTES.

Order 32, No. 8. The abstract definition is $s_1^4=s_2^2=(s_1s_2)^4=s_1^2s_2s_1s_2s_1^2s_2s_1^3s_2=1$. s_1^2, $(s_1s_2)^2$ and $(s_2s_1)^2$ generate an abelian subgroup of order 8, type (1,1,1), which is invariant under s_1. This H_{16}, thus obtained, is invariant under s_2. Hence, G is of order 32. Its generating substitutions are $s_1=aehf\cdot cgdb$, $s_2=ac\cdot eg$.

Order 48, No. 2. This group comes under one of the generalizations of the tetrahedral group, *i. e.*, the groups generated by two operators with a common cube, whose product is of order 2.*

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* E. H. Moore, *Proc. of the London Math. Soc.*, Vol. XXVIII (1897), pp. 357–366.

Order 48, No. 3. $s_1^3=s_2^3=s_3^3=(s_1s_2)^2=(s_1s_3)^2=(s_2^2s_3)^2=(s_3s_2s_1^2)^2=1$. s_1s_2, s_2s_1, s_1s_3 and $s_3^2s_1s_2^2$ generate an abelian group of order 16 and type (1, 1, 1, 1) which is invariant under the group. s_1 may be added, giving G_{48}. The group contains 32 operators of order 3 and 15 of order 2. Its generating substitutions may be chosen $s_1=abc\cdot efg$, $s_2=abd\cdot ehg$, $s_3=acb\cdot efh$.

Order 56, No. 1. $s_1^7=s_2^7=(s_1s_2)^2=s_1s_2^3s_1^4s_2^2=1$. s_2s_1, $s_2^2s_1^2$, $s_2^3s_1^3$ generate an abelian subgroup of order 8 and type (1, 1, 1) which is invariant under s_1, so that G is of order 56. $s_1=abcdefg$, $s_2=abdhcge$.

Order 60, No. 1. $s_1^2=s_2^6=(s_1s_2)^{10}=(s_1s_2^2)^2=1$. $[s_2, (s_1s_2)^2]$ is of order 30, in which $(s_1s_2)^2$ generates an invariant subgroup of order 5.

$$s_2^{-1}\cdot(s_1s_2)^2s_2=(s_2s_1)^2=(s_1s_2)^8, \quad s_2^{-1}(s_1s_2)^2s_2=(s_2s_1)^2=(s_1s_2)^8.$$

G is generated by $s_1=ad\cdot bc\cdot efg$, $s_2=ab\cdot ce\cdot fhg$.

Order 64, No. 2. This group of order 64, which comes under the theorem of paragraph 2, contains as invariant subgroups both G_{32}, No. 9, and G_{32}, No. 6. The third group of this same order is defined by extending the definition of G_{32}, No. 8. It likewise involves G_{32}, No. 7.

Order 64, No. 4. $s_1^4=s_2^4=(s_1s_2)^2=(s_1s_2^2)^4=(s_2s_1^3)^4=1$. s_1 and s_2^2 satisfy the conditions of G_{32}, No. 8. $s_1^4=(s_2^2)^2=(s_1s_2^2)^4=s_1^2s_2^2s_1s_2^2s_1^2s_2^2s_1^3s_2^2=s_1^2s_2s_1s_2s_1^3=1$. This H_{32} is invariant for $s_2^{-1}s_1s_2=s_2^2s_1$. The group is generated by $s_1=aebf\cdot cgdh$, $s_2=agce\cdot bh\cdot df$.

Order 64, No. 5. $s_1^8=s_2^4=(s_1^2s_2^2)^2=(s_1s_2)^2=1$. $[s_2, (s_1^2s_2^2)]$ is a subgroup of order 32. It is simply isomorphic with G_{32}, No. 8, above, for

$$s_2^4=(s_2^2s_1^2)^2=(s_2^3s_1^2)^4=s_2^2\cdot s_2^2s_1^2\cdot s_2\cdot s_2^2s_1^2\cdot s_2^2\cdot s_2^2s_1^2s_2^3\cdot s_2^2s_1^2=1.$$

This is invariant under s_1 for $s_1^{-1}\cdot s_2^2s_1^2\cdot s_1=s_1^7s_2^2s_1^3=s_2s_1^2s_2\cdot s_1^4$. The square of s_1 is in G_{32}. Hence, G is of order 64. Its generating substitutions may be chosen $s_1=afdechbg$, $s_2=afch\cdot be\cdot dg$.

Order 96, No. 4. $s_1^6=s_2^4=(s_1s_2^{-1})^4=(s_1s_2)^2=(s_2^2s_1^3)^2=1$. s_1^3, s_2^2, $(s_1s_2^{-1})^2$, $(s_2^{-1}s_1)^2$ are all operators of order 2 and are all commutative. The abelian group of order 16 and type (1, 1, 1, 1) generated by them is invariant under s_1^2, making a subgroup of order 48 which is invariant under s_2. $s_1=acb\cdot ef\,.\,gh$, $s_2=abcd\cdot efgh$ may be chosen as generating operators.

Order 96, No. 5. $s_1^{12}=s_2^4=(s_1^6s_2^2)^2=(s_2s_1^5)^2=(s_1s_2^2)^3s_1^9=1$. s_1^5 and $s_1s_2^2$ satisfy the conditions $t_1^{12}=t_2^{12}=(t_1t_2)^2=1$, $t_1^3=t_2^3$, and hence generate a group of order

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* G. A. Miller, "Generalization of the Groups of Genus Zero," *Trans. of the Am. Math. Soc.*, Vol. VIII (1907), p. 8.

48 which is invariant under s_2. $s_1 = abc \cdot efgh$, $s_2 = abcd \cdot fh$ are two substitutions satisfying these conditions.

Order 96, No. 6. $s_1^4 = s_2^4 = (s_1 s_2)^3 = (s_2 s_1^2)^4 = (s_1^2 s_2^2)^2 = 1$. The transforms of $s_1^2 s_2^2$, by s_1, s_2 and $s_1 s_2$, give, with $s_1^2 s_2^2$, four operators of order 2 which are all commutative. This abelian subgroup of order 16 is invariant under the group. It may be extended by $s_1 s_2$, making a subgroup of order 48 which is again invariant and may be extended by s_1. $s_1 = abcd \cdot eg$, $s_2 = ab \cdot ehfg$.

Order 96, No. 7. $s_1^6 = s_2^6 = (s_1 s_2)^2 = (s_1^3 s_2^3)^2 = (s_1 s_2^3)^3 = 1$. s_1 and $s_2^3 s_1^2$ generate a subgroup of order 24 for $s_1^6 = (s_2^3 s_1^2)^6 = (s_2^3 s_1^2 s_1)^2 = 1$ and $(s_2^3 s_1^2)^3 = s_1^3$. But the subgroup of order 24 is transformed into itself by no operator outside itself. G_{96}, however, contains G_{48}, No. 3, as an invariant subgroup. $s_1 = afcebg \cdot dh$, $s_2 = ahdgbf \cdot ce$.

Order 96, No. 8. $s_1^6 = s_2^2 s_1^3 = s_2^2 s_3^3 = s_1^2 s_3 s_1^2 s_3^4 = s_1 s_2 s_1^2 s_3 s_2 s_3^5 = 1$. s_2 and $s_1^{-1} s_2 s_1$ generate the quaternion group which is invariant under $s_3 s_2 s_1^2$. The subgroup of order 32 $[s_2, s_1^{-1} s_2 s_1, s_3 s_2 s_1^2]$ is invariant under s_1. $s_1 = aedbfc \cdot gh$, $s_2 = acbd \cdot ehfg$, $s_3 = adhbcg \cdot ef$.

Order 168, No. 1. $s_1^6 = s_2^6 = (s_1 s_2)^7 = (s_1^3 s_2^3)^2 = s_2^2 s_1^3 s_2 s_1^5 s_2^3 s_1^4 = (s_1 s_2^3)^3 = s_1^5 s_2^3 s_1 s_2^4 s_1^3 s_2^5 = 1$. s_1^3, s_2^3, $s_2^{-1} s_1^3 s_2$ generate an abelian group of order 8 and type $(1, 1, 1)$. If this is extended by $s_1 s_2$, the group generated is of order 56 and simply isomorphic with the group of order 56 defined above. Extend this H_{56} by s_1 and the order of the group is 168. It may be generated by $s_1 = ab \cdot cfgdeh$, $s_2 = agbfde \cdot ch$.

Order 180, No. 1. $s_1^{15} = s_2^{15} = s_1^5 s_2^{10} = (s_1^6 s_2^6)^2 = (s_2^6 s_1^6 \cdot s_2^3)^3 = 1$. $s_2^6 s_1^6$ and s_2^3 generate the icosahedral group which is invariant in G_{180}. s_2 and this icosahedral subgroup give the entire group. $s_1 = acebd \cdot fgh$, $s_2 = acedb \cdot fgh$.

Order 192, No. 1. $s_1^6 = s_2^4 = (s_1 s_2^{-1}) = (s_1 s_2^{-1})^4 = (s_1^3 s_2^2)^2 = s_3^2 = s_3 s_2 s_3 s_1^3 s_2 s_1^3 = (s_1 s_3)^2 = 1$. This is simply G_{96}, No. 4, extended by an operator of order 2 which is commutative with s_1 and transforms s_2 just as s_1 does.

Order 192, No. 2. $s_1^6 = s_2^4 = (s_2 s_1^2)^4 = (s_2 s_1^3)^2 = (s_2^2 s_1^2)^3 = (s_1 s_2)^4 = 1$. s_2 and $s_2 s_1^2$ generate G_{96}, No. 6. This is invariant under s_1, thus with s_1 generating the whole group. $s_1 = aebhdg \cdot ef$, $s_2 = afch \cdot bgde$.

Order 192, No. 3. $s_1^6 = s_2^6 = (s_1 s_2)^4 = (s_1^3 s_2 s_1^3 s_2^5) = 1$. $s_2 s_1$, $s_2 s_1 s_2^3$ and $s_1^{-1} s_2 s_1 s_2^3 s_1$ generate a group of order 64 which is simply isomorphic with G_{64}, No. 3. With s_2^3, this subgroup generates the entire group. $s_1 = ab \cdot cfgdeh$, $s_2 = ace \cdot bfd \cdot gh$.

Order 192, No. 4. $s_1^4 = s_2^6 = (s_1 s_2)^4 = s_1^3 s_2^3 s_1 s_2^3 = (s_1^2 s_2^2)^3 s_2^3 = s_1^3 s_2^5 s_1^2 s_2 s_1^3 s_2^5 s_1^2 s_2 = 1$. s_1^2, s_2^3 and $s_2^5 s_1^2 s_2$ generate an abelian group of order 8, type $(1, 1, 1)$, which is invariant under s_1 and s_2. The quotient group as regards this invariant subgroup is of order 24, the octahedral group. $s_1 = agce \cdot bhdf$, $s_2 = aegbfh \cdot cd$.

Order 192, No. 5. $s_1^8 = (s_1^6 s_2^6)^2 = (s_1 s_2^2)^4 = (s_1^2 s_2)^4 = (s_1 s_2)^3 = (s_1 s_2^{-1})^3 = s_1^4 s_2^4 = 1$. s_1^2, s_2^2, $(s_1 s_2^2)^2$, $(s_1^2 s_2)^2$ generate a group of order 32 which is invariant under the whole group. The quotient group as regards this invariant subgroup is the dihedral group of order 6. $s_1 = agfdbhec$, $s_2 = agdfbhce$.

Order 288, No. 1. This group is the direct product of the symmetric group of order 24 and the alternating group of order 12. It is generated by $s_1 = acbd \cdot ehg$, $s_2 = abcd \cdot efg$. $s_1^{12} = s_2^{12} = (s_1 s_2)^6 = (s_1^4 s_2 s_1 s_2)^2 = s_2^3 s_1^4 s_2^9 s_1^8 = s_1^3 s_2^4 s_1^9 s_2^8 = (s_2 s_1)^2 s_2^9 s_1^9 = (s_1^2 s_2^{-1})^2 = 1$. s_1^3 and $(s_1 s_2)^2$ generate the symmetric group of order 24 which is invariant. With respect to this, the quotient group is the tetrahedral group.

Order 288, No. 2. This is simply an extension of G_{144}, No. 1, by an operator of order 2 which transforms each generator into its inverse.

Order 288, No. 3. $s_1^6 = s_2^2 = (s_1 s_2)^6 = (s_1^3 s_2)^4 = (s_1^2 s_2)^2 (s_2 s_1^2)^2 = 1$. s_1 and $s_2 s_1 s_2$ generate a group of order 144, which is simply isomorphic with the first group of order 144 above. When this is extended by s_2, the entire group is given. $s_1 = abc \cdot ef \cdot gh$, $s_2 = ae \cdot bf \cdot cg \cdot dh$.

Order 336, No. 1. $s_1^7 = s_2^8 = (s_1^5 s_2)^6 = (s_1 s_2)^2 = (s_2 s_1^4 s_2 s_1^3)^2 = (s_2 s_1^4 s_2)^4 = 1$. s_1 and $(s_1^5 s_2)^2$ satisfy the conditions for the simple group of order 168, and it is invariant under s_2. Since s_2^2 is in this subgroup, the order of the group is 336. $s_1 = bfhgdec$, $s_2 = achfgedb$.

Order 360, No. 1. $s_1^{12} = s_2^5 = (s_1^6 s_2)^3 = (s_1^9 s_2^3)^2 = s_1^4 s_2 s_1^8 s_2^4 = 1$. s_2 and s_1^6 generate the icosahedral group, for they satisfy the conditions $t_1^2 = t_2^5 = (t_1 t_2)^3 = 1$. This is invariant under s_1 and contains the sixth power of s_1. Hence, G is of order 360. $s_1 = abcd \cdot fgh$, $s_2 = adbec$.

Order 360, No. 2. This group is simply G_{180} defined above extended by an operator of order 2 which transforms $s_1^6 = s_2^6$ into its inverse, and s_1^3 and s_2^3 into themselves. The next group of order 360 likewise involves G_{180}. The extending operator is of order 2 and transforms s_1 and s_2 into their eleventh powers.

Order 384, No. 1. This group is composed of all the substitutions on eight letters that leave invariant $ac \cdot bd \cdot eg \cdot fh$. It involves three Sylow subgroups of order 128 and sixteen of order 3. It also contains G_{192}, No. 4, and G_{192}, No. 5, invariantly. $s_1^4 = s_2^6 = (s_1 s_2)^8 = (s_2 s_1 s_2^3 s_1)^3 = (s_1 s_2^{-1})^4 = (s_1^2 s_2 s_1 s_2)^2 = 1$. $s_2 s_1$ and $s_2^3 s_1$ generate G_{192}, No. 5. $s_1 = abe \cdot cdg \cdot fh$, $s_2 = abef \cdot cdgh$.

Order 576, No. 1. This is simply the product of two octahedral groups. No. 2 of the same order is G_{288}, No. 8, extended by an operator of order 2 which is commutative with s_2 and transforms s_1 into its inverse.

Order 576, No. 3. $s_1^6 = s_2^4 = (s_1 s_2^2)^2 = (s_1^3 s_2^3)^4 = (s_2^3 s_1 s_2 s_1)^3 = 1$. This group involves G_{144}, No. 1, invariantly and is generated by G_{144} $[s_1, s_2^3 s_1 s_2]$ and s_2. $s_1 = abc \cdot ef \cdot gh$, $s_2 = aebf \cdot cg \cdot dh$.

Order 720, No. 1. $s_1^{12} = s_2^{16} = (s_1^6 s_2^2)^3 = (s_1^9 s_2^6)^2 = (s_1 s_2)^2 = s_1^{11} s_2^5 = s_2^5 s_1^7$. $s_1^4 s_2^2 = s_2^2 s_1^4$. s_1^3 and s_2^2 generate the symmetric group of order 120 which is invariant under s_1 and s_2. If G_{120} is extended by s_1, the resulting group of order 360 is likewise invariant under s_2. $s_1 = abcd \cdot fgh$, $s_2 = aedcb \cdot fh$.

Order 1152, No. 1. $s_1^{12} = s_2^{12} = s_1^4 s_2 s_1^4 s_2 s_1^8 s_2 s_1^8 s_2 = (s_1^3 s_2)^2 (s_1^9 s_2)^2 = (s_1^3 s_2 s_1^4 s_2)^2 = 1$. s_1 and $s_2 s_1 s_2$ satisfy the defining conditions given above for G_{576}, No. 1. This G_{576} is invariant under s_2. $s_1 = abcd \cdot efg$, $s_2 = ae \cdot bf \cdot cg \cdot dh$.

Order 1344, No. 1. This group is the holomorph of the abelian group of order 8 and type (1, 1, 1). It is generated by $s_1 = bfhgdec$, $s_2 = ahcdbfg$, $s_3 = abf \cdot cdh$.

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